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Generalization of Supersymmetric Quantum Mechanics

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A generalization of supersymmetric quantum mechanics can be obtained in two different ways using the theory of the q-deformation of the oscillator algebra, according to whether q is a root of unity or not. In the first case the fractional supersymmetric quantum mechanic is between bosons and q-bosons. In the second case we obtain the deformed supersymmetric quantum mechanics by considering bosons and a deformed truncated oscillator algebra.

The theory of the deformation of Lie algebras has received wide attention. Most of the deformed physical Lie algebras (also called quantum groups) has been realized starting from the deformation of the Heisenberg algebra. In this sense quantum groups can be formulated such that the defining algebraic relations are expressed as generalized commutation ones. Indeed they include phases in quadratic equations instead of just plus or minus signs, so they may be viewed as a natural generalization of supersymmetry. It has been discovered (Biedenharn, 1989; Macfarlane, 1989) that the q-deformation of the oscillator algebra provides a new algebra interpolating between the ones describing Bose and Fermi statistics. Moroever, an interesting connection between the latter and the fractional supersymmetry (F-Susy) has been established and has been seen as a generalization of supersymmetry (Leclair and Vafa, 1993). Another elegant way leading to generalized supersymmetry has been discussed by Rubakov and Spirodonov (1988), who constructed parasupersymmetric quantum mechanics (PSQM) by considering one boson and one parafermion of order 2.
The corresponding supercharges Q, Q^+ and the Hamiltonian operator H satisfy

$$Q^3 = Q^{+3} = 0$$

$$Q^2 Q^+ + Q Q^+ Q + Q^+ Q^2 = 4QH$$

$$[H, Q] = 0, \qquad [H, Q^+] = 0 \tag{1}$$

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Bagchi *et al.* (1995) generalized this result to an arbitrary order p, realizing parasupersymmetry (parasusy) using ordinary bosons and those corresponding to the truncated harmonic oscillators at an arbitrary order p. Moreover they showed that even though the parasusy algebras are different from the usual parasupersymmetric quantum machanics (PSQM), the consequences of the two are identical.

In this paper we propose a new generalization of supersymmetric quantum mechanics (SQM) in two different ways. The first one involves bosons and q-deformed bosons with q a root of unity. In this case, we obtain fractional supersymmetric quantum mechanics (FSQM). The second way uses bosons and a q-deformed truncated oscillator (with q being generic) instead of fermions to construct the ordinary supersymmetry.

To start, let us introduce the q-deformed oscillator algebra defined by considering the creation and annihilation operators a, a^+ and N satisfying

$$a\,a^+ - q\,a^+a = 1$$
$$[N, a] = -a, \qquad [N, a^+] = a^+ \tag{2}$$

with the condition

$$q^k = 1, \qquad \text{where} \quad k \quad \text{is an integer number and} \quad k \geq 2 \tag{3}$$

The elements a and a^+ are also called quonic operators in the literature and interpolate between the bosonic and fermonic ones for particular values of k. The corresponding statistics are exotic. We suppose that these exotic particles (quons) obey the generalized Pauli exclusion principle stating that no more than $k-1$ identical particles of fractional spin $s = k^{-1}$ can live in the same quantum state. The relevant Fock space is generated by the set of eigenvectors

$$F = \{|n\rangle, n = 0, 1, \ldots, k-1\} \tag{4}$$

The actions of operators a, a^+ and N in this basis are given by

$$a|n\rangle = [n]|n-1\rangle \tag{5a}$$
$$a^+|n\rangle = |n+1\rangle \tag{5b}$$
$$N|n\rangle = n|n\rangle \tag{5c}$$

where

$$[x] = \frac{q^x - 1}{q - 1}$$

The creation and annihilation operators satisfy the following nilpotency condition:

$$a^k = a^{+k} = 0 \tag{6}$$

which means that the q-deformed oscillator (quons) interpolates between the Pauli matrices ($k = 2$) and the usual bosonic oscillator ($k = \infty$).

In the following we will deal with these basic tools to formulate FSQM, so we will consider ordinary bosons and quons. Before doing this, we present the necessary polynomial commutation relation:

$$\begin{aligned} &a^{k-1}a^+ + a^{k-2}a^+a + \ldots + aa^+a^{k-2} + a^+ a^{k-1} \\ &= ([1] + \ldots + [k-1])a^{k-2} \end{aligned} \tag{7}$$

This relation is obtained by using the q-commutation relation (2). It is easy to see that the coefficient appearing in the right-hand side of equation (7) can be expressed as

$$[1] + [2] + \ldots + [k-1] = \sum_{i=1}^{k-1} [i] = \sum_{i=1}^{k-1} \frac{q^i - 1}{q - 1} = \frac{k}{1 - q} \tag{8}$$

Now we can introduce the notion of the FSQM using the equality (7). In the above Fock space the operators a and a^+ can be viewed as $k \times k$ matrices given by

$$(a)_{\alpha\beta} = [\alpha - 1]\, \delta_{\alpha+1,\beta} \tag{9a}$$

$$(a^+)_{\alpha\beta} = \delta_{\alpha,\beta+1} \tag{9b}$$

where $\alpha,\beta = 1, 2, \ldots, k$.

The supercharge operators are defined by

$$(Q)_{\alpha\beta} = [\alpha - 1]\,(P + iW_\alpha)\, \delta_{\alpha+1,\beta} \equiv (b^+ a)_{\alpha\beta} \tag{10a}$$

$$(Q^+)_{\alpha\beta} = (P - iW_\beta)\, \delta_{\alpha,\beta+1} \equiv (ba^+)_{\alpha\beta} \tag{10b}$$

Here P and W_α, $\alpha = 1, \ldots, k$, are, respectively, the translation and superpotential operators, and b and b^+ are the annihilation and creation bosonic operators ($bb^+ - b^+b = 1$) commuting with a and a^+. We recall that b and b^+ act on a bosonic Fock space F_B, $F_B = \{|n\rangle, n = 0, 1, \ldots, \infty\}$.

Using these tools, one can easily see that

$$Q^k = Q^{+k} = 0 \tag{11}$$

We note that the equalities (11) are equivalent to the formula (6).

The general form of the Hamiltonian is given by

$$H_{\alpha\beta} = H_\alpha \delta_{\alpha\beta}$$

and

$$H_r = \frac{P^2}{2} + \frac{1}{2}(w_r^2 - w_r') + \frac{1}{2}C_r \tag{12a}$$

$$H_k = \frac{P^2}{2} + \frac{1}{2}(w_k^2 + w_k') + \frac{1}{2}C_k \tag{12b}$$

where $r = 1, 2, \ldots, k - 1$.

One can verify that this Hamiltonian operator commutes with the supercharges Q and Q^+ if the following equality holds:

$$w_{s-1}^2 + w_{s-1}' + C_{s-1} = w_s^2 - w_s' + C_s \tag{13}$$

The constants C_i may be interpreted physically as the quonic oscillator energy.

Now we give the polynomial relation between the supercharges Q and Q^+; these relations are obtained by a direct calculus if one uses the equalities (7) and (10). We have

$$\begin{aligned} &Q^{k-1}Q^+ + Q^{k-2}Q^+ Q + \ldots + Q^+Q^{k-1} \\ &\quad = 2([1] + [2] + \ldots + [k-1])\, Q^{k-2}\, H \end{aligned} \tag{14}$$

To obtain the relation (14), it is necessary to have the following condition on the constants C_i:

$$C_1 + 2q^{-1}C_2 + \ldots + (k-1)q^{2-k}\, C_{k-1} = 0 \tag{15}$$

Note that the quonic oscillators introduced above for the realization of FSQM are qualitatively different from the case where q is generic. In fact, in the first case the Hilbert space is finite dimentional, while in the second case the Fock space is an infinite-dimentional space. This constitutes a problem in the sense that supersymmetry cannot be generalized if one considers an infinite-dimentional Fock space. To remove this difficulty, we propose the truncated oscillator introduced by

$$aa^+ - qa^+a = 1 - [l]K, \qquad Ka = 0 \tag{16}$$

where K is a projection operator defined by $K \equiv |l-1\rangle\langle l-1|$ and l is an arbitrary integer.

The Fock space corresponding to this truncated oscillator is given by

$$F = \{|n\rangle, n = 0, 1, \ldots, l-1\} \tag{17}$$

We point out that the deformation parameter q and the integer l appearing in this Fock space are independent, in contrast to the first case ($q^k = 1$) (FSQM).

The action of the operators a and a^+ on F is as follows:

$$a|n\rangle = [n]|n-1\rangle \tag{18a}$$

$$a^+|n\rangle = \theta(l-1-n)|n+1\rangle \tag{18b}$$

$$N|n\rangle = n|n\rangle \tag{18c}$$

$$K|n\rangle = \delta_{l,n+1}|n\rangle \tag{18d}$$

$\theta(x)$ is a step function:

$$\theta(x) \equiv \begin{cases} 0 & \text{if} \quad x \leq 0 \\ 1 & \text{if} \quad x > 0 \end{cases} \tag{19}$$

By a direct calculus one can verify the nilpotency condition on a and a^+ in this matricial representation.

As in the first case, we give the nontrivial polynomial commutation relation between a and a^+ as

$$a^{l-1}a^+ + a^{l-2}a^+a + \ldots + aa^+a^{l-2} + a^+a^{l-1} = ([1] + \ldots + [l-1])a^{l-2} \tag{20}$$

We note that, by taking the classical limit q going to one, we recover the result obtained in Khare (1992, 1993).

One can extract the polynomial commutation relations in terms of supercharges Q and Q^+ as follows:

$$Q^{l-1}Q^+ + Q^{l-2}Q^+Q + \ldots + Q^+Q^{l-1} = 2(1 + [2] + \ldots + [l-1])Q^{l-2}H \tag{21}$$

The nilpotency condition on Q and Q^+ is immediate from those on a and a^+ and the relation satisfied by the constants C^s remains the same as in equations (15); one has to substitute k by l:

$$C_1 + 2q^{-1}C_2 + \ldots + (l-1)q^{2-l}C_{l-1} = 0 \tag{22}$$

It is obvious that the relation (21) can be considered as a deformation of the equality

$$Q^{l-1}Q^+ + Q^{l-2}Q^+Q + \ldots + Q^+Q^{l-1} = l(l-1)Q^{l-2}H \tag{23}$$

obtained in Khare (1992, 1993). We find the result given by Rubakov and Spiridonov (1988) if one considers $l = 3$ and $q = 1$. The ordinary supersymmetry corresponds to the case $l = 2$ and $q = 1$.

To summarize, in this work we have constructed the fractional supersymmetric quantum mechanics (FSQM) of an arbitrary order k. This construction has been based on the notion of the deformed oscillator. We showed also that the Hamiltonian operator has a nontrivial form in terms of the supercharges. The ordinary supersymmetric quantum mechanics is present for

$k = 2$, so the FSQM seems to be its consistent generalization. In another way, we also showed that the supersymmetric quantum mechanics is between bosons and the q-truncated oscillator and we presented the corresponding Hamiltonian.

To end this paper, we remark that this result (FSQM) can be found starting from the para-Grassman differential calculus (Filippov *et al.*, 1992). The latter is based on the relation between variables and derivatives $(\theta, \partial_\theta)$ as

$$\partial_\theta \, \theta - q\theta \, \partial_\theta = 1 \tag{24}$$

with the condition

$$\theta^k = \partial_\theta^k = 0 \tag{25}$$

The para-Grassmanian variables are represented by $k \times k$ matrices in this work; one can construct the fractional supersymmetric quantum mechanics by using this formalism. Indeed they are expressed by (Filippov *et al.*, 1992)

$$\begin{aligned} \langle m|\theta|n\rangle &\sim \delta_{m,n+1} \\ \langle m|\partial_\theta|n\rangle &\sim \delta_{m,n-1} \end{aligned} \tag{26}$$

This matricial representation of θ and ∂_θ allows us to generalize the supersymmetry in a way similar to the one developed in (Leclair and Vafa, 1993). More details of this work will be clarified in a further paper (Hassouni and Daoud, n.d.).

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